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OPTIMAL DESIGN VIA VARIATIONAL PRINCIPLES: THE ONE-DIMENSIONAL CASE

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ABSTRACT. – By reformulating a typical optimal design problem in a variational format, we are able to prove existence results in a rather general framework allowing non-linear state equations and costs functional depending on derivatives of states. In this work, we restrict attention to the one-dimensional situation. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

We would like to consider optimal control problems goberned by state equations which together with boundary conditions are in fact boundary value problems. In dimension greater than one, these are typically elliptic partial differential equations of second order. The distinguished feature we would like to focus on is that controls act on the principal structural part of the state equation, so that these problems are usually refer to in the literature as optimal design problems, structural optimization problems, etc. What we are interested in is to consistently explore the possibility of recasting some of these situations in a purely variational format so as to avoid the non-local character of the state equation, and see how far one can go in proving existence results or analyzing non-existence situations. In this paper we will restrict attention to dimension one, and subsequent papers will address the same type of questions in higher dimensions. As a matter of fact, for some specific situations this point of view has already been studied in [9,10]. The literature on optimal design, structural optimization, optimal control, shape optimization, ... is huge. Our list of references focuses on a few papers directly related to our purposes.

The concrete description of the family of problems we want to analyze incorporates four main ingredients: three functions F, G, V, and a set K. Specifically, let K be a preassigned subset of $\mathbf{R}^n$, and

$$F(x,u,y,\lambda):(0,1)\times K\times\mathbf{R}^m\times\mathbf{R}^m\to\mathbf{R}^*,$$
$$G(x,u,y,\lambda):(0,1)\times K\times\mathbf{R}^m\times\mathbf{R}^m\to\mathbf{R}^l,$$
$$V(x,u,y,\lambda):(0,1)\times K\times\mathbf{R}^m\times\mathbf{R}^m\to\mathbf{R}^d,$$

three Carathéodory functions. This last requirement (being a Carathéodory function) imposes a measurable dependence on $x\in(0,1)$ and continuity with respect to all other variables. We have, without loss of generality, assumed that our one-dimensional domain is the unit interval $(0,1)$. Here $\mathbf{R}^* = \mathbf{R} \cup \{+\infty\}$, and since F will be an integrand associated to some cost, we assume it is bounded below. More assumptions will be added as we proceed in our analysis. The general problem, referred to as (P) throughout this paper, is:

Find an optimal pair $(u(x), y(x))$, belonging to the admissible set of pairs verifying:

$$u(x) \in \mathcal{U} = \{v \in L^\infty(0,1):\ v(x) \in K,\ \text{a.e. } x \in (0,1)\},$$

$$\begin{cases} -[G(x, u(x), y(x), y'(x))]' = 0, & \text{in } (0,1), \\ y(0) = y_0, \quad y(1) = y_1, \end{cases}$$

$$\int_0^1 V\big(x, u(x), y(x), y'(x)\big)\,\mathrm{d}x \leqslant \gamma,$$

which minimizes the cost

$$I(u, y) = \int_0^1 F\big(x, u(x), y(x), y'(x)\big)\,\mathrm{d}x.$$

The boundary data y_0, y_1 and the vector $\gamma \in \mathbf{R}^d$ are part of the data set. Other types of boundary conditions can be allowed. For the sake of notational simplicity, we will not indicate the target space for functions in spaces as it will always be understood by the context and no confusion will arise. As is usual in control theory, we identify u as controls and y as associated states.

From the point of view of applications, one of the most common and relevant features of these problems is related to non-existence of optimal solutions. Since the pioneering paper [3], many different situations have been studied. Homogenization theory (see for instance [4–7,11–13]) has been the main underlying tool to deeply comprehend, both analytically and computationally, this lack of existence of optimal solutions in some situations. Due to the peculiarity of dimension one, in some of these instances existence can be achieved by special methods which cannot be applied in higher dimensions. Our goal in this work is to understand the interplay among the functions F, G and V, and the set K with respect to the issue of existence-nonexistence of optimal solutions. We would like to emphasize that the main motivation for our analysis is the great generality it allows that can materialize in the following two points

1. explicit dependence on derivatives of states in the cost functional;
2. nonlinear state equations.

As pointed out above, the main tool in our analysis consists of reformulating the problem in such a way that the non-local character of the state equation, which is one of the principal difficulties in studying this sort of problems, is substituted by the introduction of an appropriate new independent variable in the form of a potential. As a result, we have an equivalent, local, variational problem with more independent variables. The whole point is then to apply existence results, or rather existence techniques, for variational problems to this reformulation.

It might be convenient to explain how this conversion can be accomplished in a particular simple example so that readers may see what we mean by this reformulation and what the new problem looks like. For simplicity, we ignore the constraint coming from the function V in this first example. Consider the following problem:

Find an optimal pair $(u(x), y(x))$, belonging to the admissible set of pairs verifying:

$$u(x) \in \mathcal{U} = \{v \in L^\infty(0,1):\ v(x) \in [\alpha, \beta],\ \text{a.e. } x \in (0,1)\},$$

$$\begin{cases} -[u(x)y'(x)]' = p(x), & \text{in } (0,1), \\ y(0) = y_0, \quad y(1) = y_1, \end{cases}$$

where $p \in L^2(0,1)$ is given, which minimizes the cost

$$I(u,y) = \int_0^1 \left|y'(x)\right|^2 \mathrm{d}x.$$

In order to reformulate this problem we proceed in several steps:

1. Auxiliary function: let P be a primitive of p

$$P(x) = \int_0^x p(s)\,\mathrm{d}s.$$

2. Transforming the state equation: the state equation can be rewritten by using P as

$$-\left[u(x)y'(x) + P(x)\right]' = 0,$$

   so that there must exist a constant c such that

$$u(x)y'(x) + P(x) = c.$$

   This identity is used to avoid the control u, and look at the pairs (c, y') as new "independent variables".

3. New integrand: we define a new density by putting:

$$\varphi(x,c,\xi) = \begin{cases} \xi^2, & (x,c,\xi) \in \Delta, \\ +\infty, & \text{else}, \end{cases}$$

   where

$$\Delta = \{(x, u\xi + P(x), \xi)\colon\ x \in (0,1), u \in [\alpha,\beta], \xi \in \mathbf{R}\},$$

   that is to say Δ is the set of triplets (x,c,ξ) so that there exists $u \in [\alpha,\beta]$ with $u\xi + P(x) = c$.

It is not hard to realize that our initial optimal control problem is equivalent to the variational problem

$$\inf_{(c,y)} I(c,y) = \int_0^1 \varphi\bigl(x,c,y'(x)\bigr)\,\mathrm{d}x,$$

where pairs $(c,y) \in \mathbf{R} \times H_0^1(0,1)$.

We believe this simple example already conveys the main idea. In our view, three are the main advantages of this approach:

1. admissible pairs (c,y) are independent;
2. there is no non-locality involved;
3. the equivalent optimization problem is a typical example in the calculus of variations, and therefore we hope to be able to analyze it in this context.

Our task consists in formalizing this perspective rigorously in more complex situations, and afterwards examine the resulting variational problem so as to prove existence results or understanding non-existence situations. One of the main technical difficulties we will encounter, is that integrands for reformulated problems are not Carathéodory functions since, in particular, they do take the value $+\infty$ abruptly.

Our main existence result requires the following notation:

$$K(x,c,y,\xi)=\bigl\{u\in K\colon\ G(x,u,y,\xi)=c\bigr\}\subset K,$$

$$\tilde F(x,c,y,\xi)=\min_{K(x,c,y,\xi)}F(x,u,y,\xi),\qquad \tilde V(x,c,y,\xi)=\min_{K(x,c,y,\xi)}V(x,u,y,\xi),$$

$$\Delta=\bigl\{(x,c,y,\xi):K(x,c,y,\xi)\neq\emptyset\bigr\},$$

$$\Delta(x)=\bigl\{(c,y,\xi):(x,c,y,\xi)\in\Delta\bigr\},$$

$$\Delta(x,c,y)=\bigl\{\xi:(x,c,y,\xi)\in\Delta\bigr\}.$$

Notice that

$$\Delta=\{\tilde F<+\infty\}=\{\tilde V<+\infty\}.$$

It is also important to realize that the definition of $\tilde F$ and $\tilde V$ require, as we have written down, that the corresponding infimum is attained. This is guaranteed either if K is bounded, or else if F and V are coercive over K in the sense that for fixed x, y and ξ,

$$\lim_{u\to\infty,u\in K}F(x,u,y,\xi)=+\infty,$$

and the same for V. This type of assumption on F, G and K is required throughout this paper.

THEOREM 1.1. – *Assume that F, G, V, and K are given as above. Suppose, in addition, that*:

(1) *$\Delta(x)$ is closed for a.e. $x\in(0,1)$*;

(2) *$\tilde F:\Delta\to\mathbf{R}$, $\tilde V:\Delta\to\mathbf{R}$ are Carathéodory functions*;

(3) *$\Delta(x,c,y)$ is convex and*

$$\tilde F(x,c,y,\cdot):\Delta(x,c,y)\to\mathbf{R},\qquad \tilde V(x,c,y,\cdot):\Delta(x,c,y)\to\mathbf{R},$$

are convex for a.e. $x\in(0,1)$ and all (c,y);

(4) *for every (x,c,y,ξ) the intersection*

$$\operatorname*{argmin}_{K(x,c,y,\xi)}F(x,u,y,\xi)\cap\operatorname*{argmin}_{K(x,c,y,\xi)}V(x,u,y,\xi)$$

is not empty;

(5) *coercivity*: *there exists a constant $k>0$, exponents $1<p\leqslant q$ and a function $h(x,u,y)\in L^\infty_{\mathrm{loc}}((0,1)\times K\times\mathbf{R}^m)$ such that for all (x,u,y,ξ)*:

$$k\bigl(|\xi|^p-1\bigr)\leqslant\tilde F(x,c,y,\xi),$$

$$\bigl|G(x,u,y,\xi)\bigr|\leqslant h(x,u,y)\bigl(1+|\xi|^q\bigr).$$

Then the original optimal control problem (P) *admits at least one optimal solution.*

Notice that the closeness, convexity or boundedness of K is not explicitly needed, although it is often hidden in requirements (1), (2) and (3) of the statement.

We will be proving this theorem as we proceed. We will first deal with the case where the additional constraint involving the function V is not present (Section 3), and then incorporate this natural restriction (Section 4). Several particular examples will be investigated in Section 5 where some remarks on non-existence have been included. The next section (Section 2) treats a general, formal derivation of the equivalent variational problem.

2. An equivalent formulation

Our starting point is the optimal control problem described in the Introduction:

Find an optimal pair $(u(x), y(x))$, belonging to the admissible set of pairs verifying

$$u(x) \in \mathcal{U} = \big\{u \in L^\infty(0,1)\colon\ u(x) \in K,\ \text{a.e. } x \in (0,1)\big\},$$

$$\begin{cases} -[G(x, u(x), y(x), y'(x))]' = 0, & \text{in } (0,1), \\ y(0) = y_0, \quad y(1) = y_1, \end{cases}$$

which minimizes the cost

$$I(u, y) = \int_0^1 F\big(x, u(x), y(x), y'(x)\big)\,\mathrm{d}x.$$

Here, we have already dropped the integral constraint since in a sense it is an additional condition not directly related to the state equation. In Section 4, we will incorporate this type of constraints and provide a full proof of Theorem 1.1.

Before proceeding any further, a word about the state equation

$$-\big[G\big(x, u(x), y(x), y'(x)\big)\big]' = 0, \quad \text{in } (0,1), \tag{2.1}$$

must be said. We will obviously place ourselves in a situation where this equation (system) together with feasible boundary conditions or other type of constraints, is weakly solvable in some $W^{1,p}(0,1)$, $p > 1$, for a given, admissible control u. There might be different contexts in which this solvability is correct but we will not stick to any particular situation. For instance, it is well-known that if G itself is the derivative with respect to λ of another scalar function

$$G(x, u, y, \lambda) = \frac{\partial W}{\partial \lambda}(x, u, y, \lambda),$$

where W is coercive and convex with respect to λ, then solvability of the state equation is guaranteed. In most of the typical examples this solvability will not be an issue, since state equations are very well-known elliptic, possibly nonlinear, differential equations. Notice also that uniqueness is not strictly needed, in particular since the number of equation l could be strictly less than the number of unknowns m. In our explicit examples however we will always take $l = m = 1$ so that for a given admissible control, we will have a unique associated state. In general, we will say that a pair is admissible for (P) if they are linked by the state equation (system) together with the appropriate boundary conditions.

As usual, we will say that (u_0, y_0) is optimal for (P) if it is admissible and

$$I(u_0, y_0) \leqslant I(u, y)$$

for any other admissible pair (u, y); and u is an optimal control if there exists a state y so that (u, y) is optimal for (P).

Let us now explain in general, precise terms how the whole control problem can be recast in a purely variational framework where existence theorems can be applied and proved.

Let (u, y) be an admissible pair for (P), so that (2.1) holds. There exists a constant $c \in \mathbf{R}^l$ such that:

$$G\big(x, u(x), y(x), y'(x)\big) = c, \quad \text{for a.e. } x \in (0, 1).$$

By using this identity, we define a new integrand $\tilde{F}$ by putting:

$$\tilde{F}(x, c, y, \xi) = \min_{K(x,c,y,\xi)} F(x, u, y, \xi),$$

where as in the Introduction

$$K(x, c, y, \xi) = \big\{u \in K\colon\ G(x, u, y, \xi) = c\big\}.$$

It is understood that $\tilde{F} = +\infty$ whenever $K(x, c, y, \xi)$ is empty. We now consider the following variational problem, which we denote by $(\tilde{\mathrm{P}})$:

$$\min \quad \tilde{I}(c, y) = \int_0^1 \tilde{F}\big(x, c, y(x), y'(x)\big)\,\mathrm{d}x,$$

where $y \in W^{1,p}(0, 1)$, $p > 1$, $y(0) = y_0$, $y(1) = y_1$ and $c \in \mathbf{R}^l$. We can precisely establish that (P) and $(\tilde{\mathrm{P}})$ are equivalent.

PROPOSITION 2.1. – *Let the infima for* (P) *and* $(\tilde{\mathrm{P}})$ *be denoted* m *and* $\tilde{m}$, *respectively.* (P) *and* $(\tilde{\mathrm{P}})$ *are equivalent*:

(1) *the two infima are the same*: $m = \tilde{m}$;

(2) *if* (u, y) *is an optimal pair for* (P) *then* (c, y) *is optimal for* $(\tilde{\mathrm{P}})$ *where* c *is the constant vector*

$$c = G\big(x, u(x), y(x), y'(x)\big).$$

Conversely, if (c, y) *is optimal for* $(\tilde{\mathrm{P}})$ *then there exists an admissible control* $u \in \mathcal{U}$ *such that*

$$c \equiv G\big(x, u(x), y(x), y'(x)\big)$$

and (u, y) *is optimal for* (P).

Proof. – Let (u, y) be an admissible pair for (P). There exists $c \in \mathbf{R}^l$ such that

$$c = G\big(x, u(x), y(x), y'(x)\big), \quad \text{for a.e. } x \in (0, 1).$$

It follows, by the definition of $\tilde{F}$, that

$$\tilde{F}\big(x, c, y(x), y'(x)\big) \leqslant F\big(x, u(x), y(x), y'(x)\big)$$

pointwise, and thus

$$\tilde{I}(c, y) \leqslant I(u, y).$$

Therefore $\tilde{m} \leqslant m$.

Conversely, let (c, y) be such that:

$$\tilde{I}(c, y) < +\infty,$$

and consider the multifunction

$$U(x) = \operatorname*{argmin}_{K(x,c,y(x),y'(x))} F\big(x, u, y(x), y'(x)\big)$$

yielding the set of $u \in K(x, c, y(x), y'(x))$ for which the minimum of $F(x, \cdot, y(x), y'(x))$ is attained. Since both F and G are Carathéodory functions, U takes on closed values (each $U(x)$ is a closed set), and there exists a measurable selection ([2]) $u(x) \in U(x)$. This implies that

$$c = G\big(x, u(x), y(x), y'(x)\big),$$

$$\tilde{F}\big(x, c, y(x), y'(x)\big) = F\big(x, u(x), y(x), y'(x)\big),$$

for a.e. $x \in (0, 1)$. Therefore (u, y) is admissible for (P),

$$I(u, y) = \tilde{I}(c, y)$$

and $m = \tilde{m}$.

The second part of the proposition is straightforward from the previous discussion. □

3. Existence without integral constraints

According to the conclusion of the previous section, existence of optimal solutions for (P) can be examined by looking at the existence issue for ($\tilde{\text{P}}$). This is the main objective of this section. Let us remind readers that the variational problem we are concerned with is:

$$\min_{(c,y)} \int_0^1 \tilde{F}\big(x, c, y(x), y'(x)\big)\,\mathrm{d}x,$$

where $c \in \mathbf{R}^l$ is a constant vector, and $y \in W^{1,p}(0, 1)$, $y(0) = y_0$, $y(1) = y_1$. Our aim is to prove existence of optimal solutions for this variational problem through the direct method by using weak lower semicontinuity. This will lead us to make the appropriate assumptions. We recall the notation introduced earlier:

$$K(x, c, y, \xi) = \big\{u \in K\colon\ G(x, u, y, \xi) = c\big\} \subset K,$$

$$\tilde{F}(x, c, y, \xi) = \min_{K(x,c,y,\xi)} F(x, u, y, \xi),$$

$$\Delta = \big\{(x, c, y, \xi)\colon\ K(x, c, y, \xi) \neq \emptyset\big\},$$

$$\Delta(x) = \big\{(c, y, \xi)\colon\ (x, c, y, \xi) \in \Delta\big\},$$

$$\Delta(x, c, y) = \big\{\xi\colon\ (x, c, y, \xi) \in \Delta\big\}.$$

THEOREM 3.1. – *Assume that F, G, and K are as above. Suppose, in addition*:
(1) $\Delta(x)$ *is closed for a.e.* $x \in (0,1)$;
(2) $\tilde{F} : \Delta \to \mathbf{R}$ *is a Carathéodory function*;
(3) $\Delta(x,c,y)$ *is convex and*

$$\tilde{F}(x,c,y,\cdot) : \Delta(x,c,y) \to \mathbf{R}$$

is convex for a.e. $x \in (0,1)$ *and all* (c,y);
(4) *coercivity*: *there exists a constant* $k > 0$, *exponents* $1 < p \leqslant q$ *and a function* $h(x,u,y) \in L^{\infty}_{\mathrm{loc}}((0,1) \times K \times \mathbf{R}^m)$ *such that for all* (x,u,y,ξ)

$$k\big(|\xi|^p - 1\big) \leqslant \tilde{F}(x,c,y,\xi),$$

$$\big|G(x,u,y,\xi)\big| \leqslant h(x,u,y)\big(1 + |\xi|^q\big).$$

Then the optimal control problem (P) *admits at least one optimal solution.*

Proof. – As usual when applying the direct method to some situation, we proceed in two steps: weak lower semicontinuity and coercivity.

Assume that $c_j \to c$ in $\mathbf{R}^l$ and $y_j \rightharpoonup y$ in $W^{1,p}(0,1)$. We would like to show

$$\int_0^1 \tilde{F}\big(x,c,y(x),y'(x)\big)\,\mathrm{d}x \leqslant \liminf_{j\to\infty} \int_0^1 \tilde{F}\big(x,c_j,y_j(x),y_j'(x)\big)\,\mathrm{d}x.$$

Without loss of generality we may assume that (c_j, y_j) is a sequence so that

$$\int_0^1 \tilde{F}\big(x,c_j,y_j(x),y_j'(x)\big)\,\mathrm{d}x < +\infty.$$

This certainly implies that:

$$\big(x,c_j,y_j(x),y_j'(x)\big) \in \Delta \quad \text{and} \quad \big(c_j,y_j(x),y_j'(x)\big) \in \Delta(x) \quad \text{for a.e. } x \in (0,1).$$

Let $\{\mu_x\}_{x\in(0,1)}$ be the Young measure associated (possibly with a subsequence) with $\{(c_j, y_j, y_j')\}$ ([8]). Because each $\Delta(x)$ is closed,

$$\operatorname{supp}(\mu_x) \subset \Delta(x).$$

On the other hand, the strong convergences $c_j \to c$, $y_j \to y$ (due to the Sobolev compactness inclussion) implies ([8]):

$$\mu_x = \delta_c \otimes \delta_{y(x)} \otimes \nu_x, \quad \text{for a.e. } x \in (0,1),$$

where $\{\nu_x\}_{x\in(0,1)}$ is the Young measure associated with $\{y_j'(x)\}$. We can therefore conclude that:

$$\operatorname{supp}(\nu_x) \subset \Delta\big(x,c,y(x)\big) \quad \text{for a.e. } x \in (0,1).$$

Since $\tilde{F}$ is a Carathéodory function when restricted to Δ and it is convex with respect to ξ, by the inequality representation in terms of Young measures ([8]) and Jensen's inequality, we can write:

$$\lim_{j\to\infty}\int_0^1 \tilde F\bigl(x,c_j,y_j(x),y_j'(x)\bigr)\,\mathrm{d}x \geqslant \int_0^1\int_{\mathbf{R}^m} \tilde F\bigl(x,c,y(x),\xi\bigr)\,\mathrm{d}\nu_x(\xi)\,\mathrm{d}x$$
$$\geqslant \int_0^1 \tilde F\bigl(x,c,y(x),y'(x)\bigr)\,\mathrm{d}x,$$

because

$$y'(x)=\int_{\mathbf{R}^m}\xi\,\mathrm{d}\nu_x(\xi).$$

This is the weak lower semicontinuity fact.

The second part of the proof consists in showing that minimizing sequences converge weakly to minimizers. This step typically requires coercivity. Assume the sequence of pairs $\{(c_j,y_j)\}$ is minimizing so that:

$$I(c_j,y_j)\searrow m,$$

where m is the value of the infimum of our variational problem. Then, by hypothesis, $\{|y_j'|^p\}$ is uniformly bounded in $L^p(0,1)$ and since $p>1$ and due to the boundary conditions, a subsequence (still denoted the same way) will converge weakly in $W^{1,p}(0,1)$ to some admissible y. On the other hand there exists a sequence of controls $\{u_j\}$ so that:

$$c_j=G\bigl(x,u_j(x),y_j(x),y_j'(x)\bigr),\quad \text{for a.e. } x\in(0,1).$$

By our coercivity assumption, and any measurable subset $E\subset(0,1)$ where $\{u_j\}$ and $\{y_j\}$ are uniformly bounded,

$$|c_j|=\frac{1}{|E|}\int_E \bigl|G\bigl(x,u_j(x),y_j(x),y_j'(x)\bigr)\bigr|\,\mathrm{d}x\leqslant C\int_0^1\bigl(1+\bigl|y_j'(x)\bigr|^q\bigr)\,\mathrm{d}x.$$

Since $q\geqslant p$, we conclude that $\{c_j\}$ is a bounded sequence of numbers, and up to some appropriate subsequence, it converges to c. By step 1, the pair (c,y) is optimal for the variational problem, and consequently there are optimal solutions for the original control problem (P).

4. A general existence theorem

In this section, we briefly incorporate to our optimization problem the integral constraint:

$$\int_0^1 V\bigl(x,u(x),y(x),y'(x)\bigr)\,\mathrm{d}x\leqslant\gamma.$$

Just as we did with the integrand F, we can define a new integrand $\tilde V$ by setting:

$$\tilde V(x,c,y,\xi)=\min_{K(x,c,y,\xi)} V(x,u,y,\xi),$$

which again is finite over the set Δ. We therefore have to suppose as a main assumption:

$$\tilde{V} : \Delta \to \mathbf{R}$$

is a Carathéodory function. One would immediately be tempted to think plausible that the new optimization problem under this new additional restriction would be equivalent to

$$\min_{(c,y)} \int_0^1 \tilde{F}\big(x, c, y(x), y'(x)\big)\, \mathrm{d}x,$$

subject to

$$\int_0^1 \tilde{V}\big(x, c, y(x), y'(x)\big)\, \mathrm{d}x \leqslant \gamma.$$

However, this is not so in general because the fact

$$F\big(x, u(x), y(x), y'(x)\big) = \tilde{F}\big(x, c, y(x), y'(x)\big)$$

does not imply by itself that the same is true for V and the same admissible control $u(x)$. We would have in general

$$V\big(x, u(x), y(x), y'(x)\big) \geqslant \tilde{V}\big(x, c, y(x), y'(x)\big),$$

but obviously this inequality is not enough to ensure

$$\int_0^1 V\big(x, u(x), y(x), y'(x)\big)\, \mathrm{d}x \leqslant \gamma$$

from

$$\int_0^1 \tilde{V}\big(x, c, y(x), y'(x)\big)\, \mathrm{d}x \leqslant \gamma.$$

Hence, we cannot be sure that u is admissible for the original optimization problem (P). A way to overcome this difficulty (probably not the only one) is to enforce that for given $(x, c, y, \xi) \in \Delta$ we can always find $u \in K$ in such a way that:

$$\tilde{F}(x, c, y, \xi) = F(x, u, y, \xi), \qquad \tilde{V}(x, c, y, \xi) = V(x, u, y, \xi),$$

hold simutaneously. For this, it suffices to have that the intersection

$$\operatorname*{argmin}_{K(x,c,y,\xi)} F(x, u, y, \xi) \cap \operatorname*{argmin}_{K(x,c,y,\xi)} V(x, u, y, \xi)$$

is never empty. These ideas would complete, together with the material in Sections 2 and 3, the proof of our main result Theorem 1.1.

5. Examples and final remarks

We would like to look at several typical situations and examples where our existence theorem can be applied, and other cases where it cannot be applied.

1. Consider the linear state equation:

$$-\big(u(x)y'(x)\big)' = p(x), \quad \text{in } (0,1),$$

$$y \in H^1(0,1), \quad y(0) = y_0, \quad y(1) = y_1,$$

where $u(x) \in K$, which is contained in $(\epsilon, +\infty)$ for some $\epsilon > 0$ and $p \in L^2(0,1)$ is fixed. Let $P(x)$ be a primitive of $p(x)$:

$$P(x) = \int_0^x p(s)\,\mathrm{d}s.$$

In this situation all dimensions are unity. It is very well-known that the state y is uniquely determined by the control u. In order to apply Theorems 1.1 or 3.1 depending on whether we have additional integral restrictions, we must care about the appropriate hypotheses in those theorems. It is elementary to find:

$$K(x,c,y,\xi) = \begin{cases} K, & \text{if } \xi = 0 \text{ and } c = P(x), \\ \dfrac{c-P(x)}{\xi}, & \text{if } \xi \neq 0 \text{ and } \dfrac{c-P(x)}{\xi} \in K, \\ \emptyset, & \text{if } \xi = 0 \text{ and } c \neq P(x) \text{ or } \xi \neq 0 \text{ and } \dfrac{c-P(x)}{\xi} \notin K, \end{cases}$$

$$\Delta = \big\{\big(x, P(x), y, 0\big)\colon x \in (0,1), y \in \mathbf{R}\big\} \cup \left\{(x,c,y,\xi)\colon \xi \neq 0, \frac{c-P(x)}{\xi} \in K\right\},$$

$$\Delta(x) = \big\{\big(P(x), y, 0\big)\colon x \in (0,1), y \in \mathbf{R}\big\} \cup \left\{(c,y,\xi)\colon \xi \neq 0, \frac{c-P(x)}{\xi} \in K\right\},$$

$$\Delta(x,c,y) = \left\{\frac{c-P(x)}{u}\colon u \in K\right\},$$

$$\tilde{F}(x,c,y,\xi) = \begin{cases} F\big(x, \frac{c-P(x)}{\xi}, y, \xi\big), & \xi \neq 0, \frac{c-P(x)}{\xi} \in K, \\ \min_K F(x,u,y,\xi), & \xi = 0, c = P(x), \\ +\infty, & \text{else.} \end{cases}$$

Therefore it is also straightforward to check:

1. $\Delta(x)$ is closed if and only if K is closed and bounded;
2. $\Delta(x,c,y)$ is convex if and only if K is convex;
3. $\tilde{F}(x,c,y,\cdot)\colon \Delta(x,c,y) \to \mathbf{R}$ is convex if and only if the function

$$\varphi(x,k,y,\xi) = F(x, k/\xi, y, \xi)$$

is convex in ξ whenever $k/\xi > 0$.

However, we may find in some situations that $\tilde{F}(x,c,y,\xi)$ is not a Carathéodory function. Indeed if we choose a sequence (c_j, y, ξ_j) such that:

$$c_j \to P(x), \qquad \xi_j \to 0, \qquad \frac{c_j - P(x)}{\xi_j} = v_j \in K \to v \in K$$

but $F(x,v,y,0)$ is not the minimum $\tilde{F}(x, P(x), y, 0)$ then

$$\tilde{F}(x, c_j, y, \xi_j) \quad \text{does not converge to} \quad \tilde{F}\big(x, P(x), y, 0\big).$$

This is actually a rather artificial or technical difficulty. From the point of view of our variational problem, consider that $\{(c_j, y_j)\}$ is a minimizing sequence for $(\tilde{\mathrm{P}})$, so that:

$$c_j \to c, \qquad y_j \rightharpoonup y \quad \text{in } H^1(0,1).$$

Suppose further that for some non-negligible set $E \subset (0,1)$ we have

$$P(x) = c, \qquad y_j'(x) \to 0, \qquad \frac{c_j - c}{y_j'(x)} = v_j(x) \in K,$$

for all $x \in E$, but $v_j(x)$ does not belong to the set where the minimum of $F(x,u,y,0)$ is taken on. In this case, we redefine:

$$\tilde{y}_j'(x) = \frac{c_j - c}{v(x)}, \quad x \in E,$$

where $v(x) \in K$ is any point where the previous minimum is attained. It is not hard to see that

$$\tilde{y}_j \rightharpoonup y,$$

and $\tilde{F}$ is "continuous along this minimizing sequence" so that the proof of Theorem 3.1 is valid for this modified sequence. Notice that even if $\tilde{y}_j$ may not be admissible concerning the appropriate values at the end-points, the weak limit is.

We can summarize all these considerations in the following corollary.

COROLLARY 5.1. – *Assume that the state equation is given as above. If K is a closed, convex, finite interval of positive numbers,*

$$\varphi(x,k,y,\xi) = F(x, k/\xi, y, \xi)$$

is convex in ξ whenever $k/\xi > 0$ and

$$c\big(|\xi|^p - 1\big) \leqslant \varphi(x,k,y,\xi), \quad c > 0, 1 < p \leqslant 2,$$

then the associated optimization problem admits optimal solutions.

Notice that the coercivity assumption concerning G is trivially satisfied in this case.

2. If we add the integral constraint

$$\int_0^1 V\big(x, u(x), y(x), y'(x)\big)\,\mathrm{d}x \leqslant \gamma$$

to the previous situation, we have to incorporate the further hypotheses of Theorem 1.1 concerning the integrand V. But because $K(x,c,y,\xi)$, when it is not empty, is either a singleton or all of K, hypothesis (4) in that theorem is correct provided that:

$$\operatorname*{argmin}_K F(x,u,y,0) \cap \operatorname*{argmin}_K V(x,u,y,0) \neq \emptyset.$$

The same considerations as above apply for the continuity issue. Therefore we have:

COROLLARY 5.2. – *Under the same state equation as above, assume that*:
(1) K *is a closed, convex, finite interval of positive numbers*;
(2) *the functions*

$$\varphi(x,k,y,\xi) = F(x,k/\xi,y,\xi), \qquad \psi(x,k,y,\xi) = V(x,k/\xi,y,\xi)$$

are convex in ξ *whenever* $k/\xi > 0$;
(3) *for a.e.* $x \in (0,1)$ *and* $y \in \mathbf{R}^m$,

$$\operatorname*{argmin}_K F(x,u,y,0) \cap \operatorname*{argmin}_K V(x,u,y,0) \neq \emptyset;$$

(4) *coercivity*:

$$c\big(|\xi|^p - 1\big) \leqslant \varphi(x,k,y,\xi), \quad c > 0,\ 1 < p \leqslant 2.$$

Then the associated optimal control problem admits optimal solutions.

Notice that the typical volume constraint condition in structural optimization problems for which

$$V(x,u,y,\xi) = u$$

stays within the range of this result. Condition (3) would require in this case that the cost integrand must be non-decreasing in the control u.

3. The next example shows that the boundedness of K is essential. Consider the problem

$$\text{Minimize} \quad \int_0^1 \big|y'(x)\big|^2 \,\mathrm{d}x$$

subject to

$$u \in L^\infty(0,1), \quad u(x) \in [1,+\infty), \text{ a.e. } x \in (0,1),$$

$$-\big(u(x)y'(x) - (1-2x)\big)' = 0, \quad y \in H_0^1(0,1).$$

We will show that the value of the infimum vanishes. Since the function vanishing identically on $(0,1)$ does not satisfy the state equation, we will have shown the non-existence of optimal solutions.

Consider the sequence:

$$y_j(x) = \frac{1}{j}\big(x - x^2\big), \quad u_j(x) = j, \quad j \geqslant 0.$$

It is easy to see that the pairs $\{(u_j, y_j)\}$ are admissible. Moreover

$$I(u_j, y_j) = \int_0^1 \left|y_j'(x)\right|^2 \mathrm{d}x = \frac{1}{j^2}\int_0^1 |1-2x|^2\,\mathrm{d}x \searrow 0,$$

as $j \to \infty$. This implies that the infimum is indeed zero.

4. For an example of a non-linear state equation we take the typical q-Laplacian in one dimension:

$$-\big(u(x)\left|y'(x)\right|^{q-1}y'(x)\big)' = p(x), \quad x \in (0,1),$$

$$y \in W^{1,q}(0,1), \quad q > 2, \quad y(0) = y_0, \quad y(1) = y_1.$$

Many of the elements we have to analyze are a direct generalization of the linear case. Namely:

$$K(x,c,y,\xi) = \begin{cases} K, & \text{if } \xi = 0 \text{ and } c = P(x),\\ \dfrac{c-P(x)}{|\xi|^{q-1}\xi}, & \text{if } \xi \neq 0 \text{ and } \dfrac{c-P(x)}{|\xi|^{q-1}\xi} \in K,\\ \emptyset, & \text{if } \xi = 0 \text{ and } c \neq P(x) \text{ or } \xi \neq 0 \text{ and } \dfrac{c-P(x)}{|\xi|^{q-1}\xi} \notin K, \end{cases}$$

$$\Delta = \big\{\big(x, P(x), y, 0\big)\colon x \in (0,1), y \in \mathbf{R}\big\} \cup \left\{(x,c,y,\xi)\colon \xi \neq 0, \frac{c-P(x)}{|\xi|^{q-1}\xi} \in K\right\},$$

$$\Delta(x) = \big\{\big(P(x), y, 0\big)\colon x \in (0,1), y \in \mathbf{R}\big\} \cup \left\{(c,y,\xi)\colon \xi \neq 0, \frac{c-P(x)}{|\xi|^{q-1}\xi} \in K\right\},$$

$$\Delta(x,c,y) = \left\{\left|\frac{c-P(x)}{u}\right|^{(2-q)/(q-1)}\frac{c-P(x)}{u}\colon u \in K\right\},$$

$$\tilde F(x,c,y,\xi) = \begin{cases} F\left(x, \dfrac{c-P(x)}{|\xi|^{q-1}\xi}, y, \xi\right), & \xi \neq 0, \dfrac{c-P(x)}{|\xi|^{q-1}\xi} \in K,\\ \min_K F(x,u,y,\xi), & \xi = 0, c = P(x),\\ +\infty, & \text{else}. \end{cases}$$

As in the linear case, we also find that it is easy to check:

1. $\Delta(x)$ is closed if and only if K is closed and bounded;
2. $\Delta(x,c,y)$ is convex if and only if K is convex;
3. $\tilde F(x,c,y,\cdot): \Delta(x,c,y) \to \mathbf{R}$ is convex if and only if the function

$$\varphi(x,k,y,\xi) = F\big(x, k/\big(|\xi|^{q-1}\xi\big), y, \xi\big)$$

is convex in ξ whenever $k/\xi > 0$.

Moreover the same observations apply concerning the continuity of $\tilde F$ with respect to (u,ξ). The analysis in the presence of an integral constraint with integrand V is also analogous.

COROLLARY 5.3. – *Assume that the state equation is given above* (q*-Laplacian*). *If*:

(1) K *is a closed, convex, finite interval of positive numbers*;

(2) *the functions*

$$\varphi(x,k,y,\xi) = F\big(x, k/\big(|\xi|^{q-1}\xi\big), y, \xi\big), \qquad \psi(x,k,y,\xi) = V\big(x, k/\big(|\xi|^{q-1}\xi\big), y, \xi\big)$$

are convex in ξ *whenever* $k/\xi > 0$;

(3) *for a.e.* $x \in (0,1)$ *and* $y \in \mathbf{R}^m$,

$$\operatorname*{argmin}_K F(x,u,y,0) \cap \operatorname*{argmin}_K V(x,u,y,0) \neq \emptyset;$$

(4) *coercivity*:

$$c\big(|\xi|^p - 1\big) \leqslant \varphi(x,k,y,\xi), \quad c > 0, 1 < p \leqslant q.$$

Then the associated optimal control problem admits optimal solutions.

5. We finally address the question of nonconvexity. One of the most typical situations in optimal design is related to the nonconvexity of the set K. Often

$$K = \{\alpha, \beta\}, \quad 0 < \alpha < \beta,$$

so that admissible controls are allowed to take on only the values α or β. For simplicity we also assume that we have a linear state equation:

$$-\big(u(x)y'(x)\big)' = p(x), \quad \text{in } (0,1),$$

$$y \in H^1(0,1), \quad y(0) = y_0, \quad y(1) = y_1.$$

We cannot apply Corollary 5.1 or 5.3. Does this mean that non-existence is to be expected? We believe this is so most of the time in higher dimensions. We will address this issue in future work ([1]). However, dimension one is somewhat special in the sense that existence can some times be easily obtained despite lack of convexity as the following simple example shows.

Under the same context as above, consider the mixed boundary conditions

$$u(1)y'(1) = \sigma, \qquad y(0) = 0,$$

and a cost functional not depending on derivatives of the state y

$$I(u) = \int_0^1 u(x)p(x)\,\mathrm{d}x + \sigma y(1).$$

This is the compliance functional yielding a measure of the work done by the loads under equilibrium. We also consider an integral constraint of the type:

$$\int_0^1 u(x)\,\mathrm{d}x = L \in [\alpha, \beta].$$

In this simplified situation the equilibrium problem can be solved in closed form:

$$y(x)=\int_0^x\left[\frac{F}{u(s)}+\frac{1}{u(s)}\int_s^1 p(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s,$$

and, consequently, the dependence of I with respect to the characteristic function χ where

$$u(x)=\chi(x)\alpha+\big(1-\chi(x)\big)\beta,$$

can be made explicit

$$I(\chi)=\int_0^1\left(F+\int_s^1 p(\tau)\,\mathrm{d}\tau\right)^2\left(\chi(s)\frac{1}{\alpha}+\big(1-\chi(s)\big)\frac{1}{\beta}\right)\mathrm{d}s.$$

It is now easy to argue that the region A where $u=\alpha$ will be that of length

$$l=\int_0^1\chi(x)\,\mathrm{d}x$$

verifying

$$\max_{s\in A}\left(F+\int_s^1 p(\tau)\,\mathrm{d}\tau\right)^2\leqslant\min_{s\in(0,1)\setminus A}\left(F+\int_s^1 p(\tau)\,\mathrm{d}\tau\right)^2.$$

By looking at the graph of the function

$$\left(F+\int_s^1 p(\tau)\,\mathrm{d}\tau\right)^2,$$

it is easy to determine the region A.

In this example, the cost does not depend on derivatives y'. We do not know what the situation is when there is some explicit dependence on derivatives of y. In general, and avoiding the difficult issue of existence under lack of convexity, the analysis may proceed as is typical in non-convex variational problems examining relaxation at different levels (see for instance [8]). Since again this will be our main concern in higher dimensions in future work, we do not pursue this issue here.

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